A Non-perturbative derivation of the exact value of the SU(2) coupling value g from the standard Electroweak Lagrangian itself.

J.R.G. (Hans) de Vries
(Dated: March 30, 2007)

The basic Electro Weak coupling constants are e, g and g'. The (zero-momentum limit) of e is know with a very high precision, with a significant improvement recently thanks to a new α value from Gabrielse/Kinoshito. Here, we first presume that the value of α, associated with vacuum polarization, consists of both a charge-term and a spin-term, and we propose a Gordon decomposition like method to subtract the spin-term from α to obtain a clean, 'charge only', value $\alpha_\emptyset$, as well as the corresponding 'clean' coupling values $e_\emptyset$, $g_\emptyset$ and $g'_\emptyset$. Within Electro weak unification, the pure SU(2) coupling g, which only relates to $W^\pm$, might be considered to be the simplest constant. We present a non-perturbative method to calculate this value from the standard electroweak Lagrangian itself, with a result which is within experimental precision. The running value of g derived in this way exhibits a high-momentum cut-off.

I. REMOVING THE SPIN CONTRIBUTION

We presume that the *spin related component* in the coupling constant e, associated with vacuum polarization, is: *An electromagnetic coupling with a magnetic moment.* So, including a first anomaly term, we make the following guess, similar to Gordon decomposition:

$$\mathrm{e} \;=\; \left(\; 1^{(charge)} + \;\; \left[\alpha\left(1+\frac{\alpha}{2\pi}\right)\right]^{(spin)} \right) \mathrm{e}_\emptyset \qquad (1)$$

Normally we have no way to determine if the 'charge only' value $e_\emptyset$ obtained in this way is physically meaningful. Our only hope is that this 'clean' value turns out to be a clean and simple mathematical value as well. Similar for example to the exact value $\frac{1}{2}$ of the proton spin resulting from an internally very complicated mix of quark spins, gluon spins, sea-quark spins plus all internal angular momenta.

It is indeed this which turns out to be the case. Using the latest Gabrielse/Kinoshita [1] value for α we obtain as the value for $\alpha_\emptyset$:

$$\begin{aligned} \alpha^{-1} &= 137.\,035\,999\,710\,(96) \\ \alpha_\emptyset^{-1} &= 139.\,045\,636\,842\,(96) \end{aligned} \qquad (2)$$

We find our hoped for 'clean and simple' mathematical value when we try:

$$\frac{2}{\pi^2}\log\,\alpha_\emptyset \;= -1.\,000\,000\,000\,26\,(14) \qquad (3)$$

A result exact to within less then two sigma of the Gabrielse/ Kinoshito result. This leads us to write:

$$\alpha_\emptyset \;=\; e^{-\frac{1}{2}\pi^2} \qquad (4)$$

The precision is so high that one is inclined to think that the simple expression (1) might be exact to all orders, in which case the observable α would have 'absorbed' any higher order terms related to the magnetic anomaly. With this presumption in mind one could in principle derive an, exact to all order, value for the observable α value with unlimited precision:

$$\alpha^{-1} \;=\; 137.\,035\,999\,528\,369\,196 \qquad (5)$$

We can now look at the remaining small error term by using this value for α for the calculation of the electron's magnetic moment with the information given in [1], and compare the result with the magnetic moment as measured by Gabrielse:

1. 001 159 652 180 85 (76)	as measured by Gabrielse.
1. 001 159 652 180 68	QED version, incl. μ and τ
1. 001 159 652 180 71	Electroweak, incl. W and Z
1. 001 159 652 182 38	Full SM version, incl. QCD

(6)

The first of the three calculated values considers only the pure QED contributions, with the inclusion of the (vacuum polarization) contributions of the muon and tau leptons. The second calculated value further includes the small electro-weak contributions, while the latter gives the full standard model result by including the hadronic contributions. There is no deviation from Gabrielse's result as long as one excludes the hadronic contributions.

II. 'CLEAN' ELECTRO-WEAK CONSTANTS

The next step is to remove the magnetic moment contributions from the electro-weak coupling constants g and g' as well. This amount to exactly the same procedure as given by expression (1). First we do need an experimental result for the Weinberg angle θ_W to relate e to g and g'. We make a decision at this point to use two different experimental average values here, both in the on-shell (s^2_W) scheme.

The first value $s^2_{W_1}$ stems from the new (Januari 2007) world average of the W-mass of 80.398(25) GeV value following the latest Tevatron CDF results [2]. The second value $s^2_{W_2}$ stems from a recent global simultaneous fit of multiple SM parameters [3]. The two values used for s^2_W are:

$$\begin{aligned} s^2_{W_1} &= 0.22264\ (51) \\ s^2_{W_2} &= 0.22306\ (33) \end{aligned} \tag{7}$$

Using the elementary relation $g = e/s_W$, see for example [4], we obtain the following values for the 'clean' values of the electro-weak coupling constant $g_\emptyset$ from $e_\emptyset$:

$$\begin{aligned} g_{\emptyset_1} &= 0.63712\ (73) \\ g_{\emptyset_2} &= 0.63652\ (47) \end{aligned} \tag{8}$$

III. NON-PERTURBATIVE QFT

We now propose a way to calculate the value g from the Weinberg-Salam model of electro-weak unification in a non-pertubative manner. The relevant part of the Electro weak Lagrangian [5] for a lepton-neutrino pair is:

$$\mathcal{L} = \bar{\chi}_L \gamma^\mu \left[i\partial_\mu - \frac{1}{2} g\, \tau \cdot \mathbf{W}_\mu \right] \chi_L \tag{9}$$

Where τ represents the weak isospin. This part of Lagrangian is very similar to the QED Lagrangian version for the electromagnetic interaction:

$$\mathcal{L} = \bar{\psi}_L \gamma^\mu \left[i\partial_\mu - eA_\mu \right] \psi_L \tag{10}$$

The electromagnetic potential field A_μ, and thus also the coupling constant e, involve an SU(2)xU(1) mixing according to electro-weak unification, while g acts on the W-bosons only.

Non-perturbatively, we replace A_μ and W_μ with (real) sinusoidal wave-functions. For instance, the time dependent part of the wave-function ψ of an electron becomes:

$$\psi(t) = u^{(s)} \exp\left(-i\frac{E}{\hbar} t + i\frac{eA_o}{\omega\hbar} \sin(\omega t) \right) \tag{11}$$

The argument represents the total phase change in time and is derived by integrating the Hamiltonian over time. This integration results in the important factor ω in the denominator. The factor $eA_o/\omega\hbar$ in front of the sinus is the quotient of two energies. eA_o represent the interaction energy of an electron in a classical electromagnetic field, while the denominator $\omega\hbar$ represents the energy from an absorption of a single photon from an electromagnetic field with a frequency ω.

An increase in the electromagnetic field strength increases the chance to absorb one (or more) photons. This quotient of energies becomes $\frac{1}{2}gW/\omega\hbar$ in the case of the electro-weak coupling constant:

$$\psi(t) = u^{(s)}_L \exp\left(-i\frac{E}{\hbar} t + i\frac{\frac{1}{2}gW}{\omega\hbar} \sin(\omega t) \right) \tag{12}$$

The sinusoidal term in the argument corresponds to *frequency modulation.* [6],[7] We can therefor use the identity:

$$e^{iQ\sin(\omega t)} = \sum_{k\,=-\infty}^{\infty} \mathrm{J}_k(Q)\ e^{ik\omega t} \tag{13}$$

Where J_k represents the Bessel J function of k^{th} order. The coefficient $J_k(Q)$ represents the amplitude for the lepton/neutrino to change its energy with an amount of $k\omega\hbar$ where k is an integer value. The expression above thus explains (semi-)classically why the total energy absorbed or emitted can only be a multiple of $\omega\hbar$!

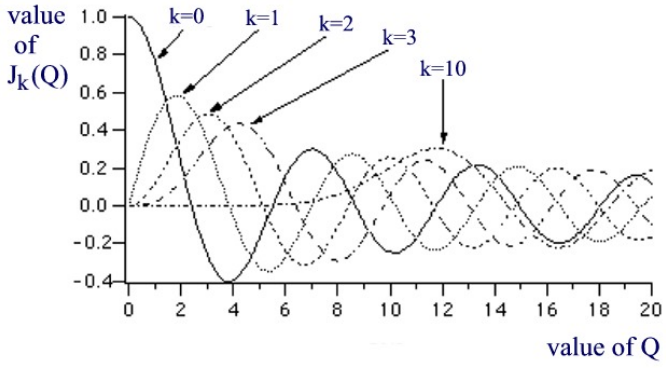

FIG. 1: Bessel J functions of k^{th} order.

The chance that *multiple* energy quanta are absorbed increases with Q, corresponding to an increase of field-strength, as well as a decrease of the frequency ω. In the

low energy limit $\omega \rightarrow 0$, the amount of absorbed quanta goes to infinity.

These Bessel coefficients have the beautiful property that they are unitary for both the amplitudes as well as the probabilities, for any value of Q:

$$\sum_{k\,=-\infty}^{\infty} \mathrm{J}_k(Q) \;= 1, \qquad \sum_{k\,=-\infty}^{\infty} \left|\mathrm{J}_k(Q)\right|^2 \;= 1 \tag{14}$$

Having an expression for the amplitude of an energy transfer of $\omega\hbar$, provides us with a way to determine the average chance of absorbing emitting a single energy quanta as a function of the energy quotient Q:

$$p(Q) \;=\; \sum_{k\,=0}^{\infty} k \left|\mathrm{J}_k(Q)\right|^2 \tag{15}$$

Summing over one half of the coefficients to get the change for either emission or absorbtion. Examining the expression numerically one finds for strong fields or for low frequency values of ω:

$$\text{if } Q \gg 1 \;\rightarrow\; p(Q) \;=\; \frac{1}{\pi}Q + \frac{1}{8\pi}Q^{-1} - \frac{1}{4}\mathrm{J}_0^2(Q) \tag{16}$$

In the low energy limit case where $\omega \rightarrow 0$, only the first term survives.

$$\lim_{Q\rightarrow\infty} \quad p(Q) \;=\; \frac{1}{\pi}Q \tag{17}$$

If we would have started our non-perturbative treatment with a coupling constant 1, then Q would be the quotient:

$$Q = \frac{W}{\omega\hbar} \tag{18}$$

In the case where the potential field energy W is N times higher as smallest energy quantum $\omega\hbar$ then we expect the chance for absorbtion or emission to be N. However, we find, in the low energy case, a value of N/π.

It follows that in a *perturbative* treatment we need to include a factor $1/\pi$ as an implicit coupling constant:

$$\frac{1}{2}g_{\emptyset} \;=\; \frac{1}{\pi} \tag{19}$$

$$g_{\emptyset} \;=\; \frac{2}{\pi} \;=\; 0.6366197723.. \tag{20}$$

A result with a precision well within one sigma for the experimental values we used:

$$\begin{aligned} g_{\emptyset_1} \;&=\; 0.63712\ (73) \\ g_{\emptyset_2} \;&=\; 0.63652\ (47) \end{aligned} \tag{21}$$

Finally, we want to look at the high energy behavior of g. We can do this analytically since we only need to consider the behavior of $\left|\mathrm{J}_k(Q)\right|^2$ near zero. We find:

$$\lim_{Q\rightarrow 0}\; p(Q) \;=\; \frac{1}{4}Q^2 \tag{22}$$

Which would lead to a running coupling constant:

$$g_{\emptyset}(Q) \;=\; \frac{1}{2}Q \;=\; \frac{1}{2}\frac{W}{\omega\hbar} \tag{23}$$

The presented method would thus lead to an inherent non-perturbtative high-energy cut-off via the coupling constant g itself.

[1] G. Gabrielse, T. Kinoshita, et al. *New Determination of the Fine Structure Constant from the Electron g Value and QED*, Physical Review letter, PRL 97,030802 (2006) http://hussle.harvard.edu/∼gabrielse/

[2] Tevatron CDF-group home page: http://www.hep.ucl.ac.uk/cdf/

[3] Jens, Erler. *Precision Electroweak Physics*, AIP Conference Proceedings – September 25, 2006 – Volume 857, pp. 85-99. preprint hep-ph/0604035

[4] Weinberg, S. *Chapter 21.3, The Electroweak Theory.* The Quantum Theory of fields, Volume II.

[5] Halzen, F. Martin, A.D. *Chapter 15.1, The Weinberg-Salem model and Beyond.* Quarks & Leptons.

[6] http://ccrma.stanford.edu/∼jos/rbeats/ Bessel_Functions.html

[7] Arfken, Weber *Chapter 11.1, Bessel Functions.* Mathematical Methods for Physicists.